Eigenvalues

Def: Let $T: V \to V$. A scalar $\lambda \in F$ is called an eigenvalue if there exists a $v \in V$, $v \neq 0$, such that $Tv = \lambda v$. A vector $v \neq 0$ with this property is called an eigenvector corresponding to the eigenvalue λ. The set of all eigenvectors of λ is called the eigenspace. $E(\lambda, T) = \ker(T - \lambda I)$

Remarks:

$Tv = \lambda v$

$Tv - \lambda v = 0$

$(T - \lambda I)v = 0$

- Eigenvalue/eigenvector realizes a "scaling". $v \mapsto \lambda v$ direction of vector v does not change.
- Many linear mappings do not have eigenvectors. e.g. rotation.

  Not true, for algebriaclly closed fields like $\mathbb{C}$. $\boxed{\mathbb{R} \text{ is not algebriaclly closed}}$

- If λ is an eigenvalue, it has many eigenvetors.

  e.g. if v is an eigenvector., then any $a \cdot v$ $(a \in K)$ is an eigenvector.

  $$T(a \cdot v) = a \cdot Tv = a \cdot \lambda v = \lambda (a \cdot v)$$

  $$\Rightarrow T(a \cdot v) = \lambda (a \cdot v)$$

- Eigenvectors corresponding to distint eigenvalues are linearly independent.

Intuition: λ_1, λ_2 two distinct eigenvalues.

i.e. $\lambda_1 \neq \lambda_2$.

Assume v_1, v_2 are eigenvectors that are not linearly independent. i.e. $v_2 = c \cdot v_1$

$$Tv_1 = \lambda_1 v_1$$

$$Tv_2 = \lambda_2 v_2 = \lambda_2 (c \cdot v_1)$$

not equal

equal

$$Tv_2 = T(c \cdot v_1) = c \cdot Tv_1 = c \cdot \lambda_1 v_1$$

contradiction!!

- Eigenvectors that correspond to the same eigenvalue do not need to be independent.

  e.g. v eigevector $\Rightarrow$ $c \cdot v$ is also an eigenvector. but v & $c \cdot v$ are not linearly independent.

  They can be linearly independent:

  e.g. $A = I$, all eigenvalues are 1.

  $I \cdot v = \underline{1} \cdot v$. But eigenvectors v can be linearly independent.

- The eigenspace $E(\lambda, T)$ is always a linear subspace of V.

Proposition: For finite-dimensional vector space, the following statements are equivalent:

(i) λ eigenvalue of T

(ii) $T - \lambda I$ not injective

(iii) $T - \lambda I$ not surjective

(iv) $T - \lambda I$ not bijective

Proposition: Suppose V is a finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $\lambda_1, \dots \lambda_m$ are distinct eigenvalues of T. Then a sum of eigenspaces $E(\lambda_1, T) + E(\lambda_2, T) + \dots + E(\lambda_m, T)$ is a direct sum. In particular

$$\dim(E(\lambda_1, T)) + \dots + \dim(E(\lambda_m, T)) \leq \dim V.$$

Theorem: Every operator $T: V \to V$ on a finite dimensional vector space with a complex field has at least one eigenvalue.

Proof: Let $n = \dim V$. Choose a vector $v \in V$, $v \neq 0$. Then the set

$$v, Tv, T^2 v, \dots T^n v$$

has to be linearly dependent. (since it consist of $n+1$ vectors in a n-dim vector space)

Find coefficients $a_0, a_1, \dots a_n$ such that

$$a_0 v + a_1 Tv + a_2 T^2 v + \dots a_n T^n v = 0$$

Now consider a polynomial on $\mathbb{C}$ with the same coefficients. $p(z) := a_0 + a_1 z + \dots a_n z^n$

Over $\mathbb{C}$, we can factorize polynomial as:

$$p(z) = c \cdot (z-\lambda_1)(z-\lambda_2) \dots (z-\lambda_m)$$

$$m \leq n$$

Consider again: $a_0 v + a_1 T v + \dots + a_n T^n v = 0$

$$\Rightarrow (a_0 + a_1 T + \dots a_n T^n) v = 0$$

$$\Rightarrow \underbrace{c \cdot (T - \lambda_1 I)(T-\lambda_2 I) \dots (T - \lambda_m I)}_{*} v = 0$$

(factorization of polynomial $\Rightarrow$ factorization of operator)

$\Rightarrow v \in \ker(*)$ (injective $\Leftrightarrow$ $\ker\{0\}$)

$\Rightarrow$ $*$ is not an injective function.

$\Rightarrow$ there exists $i \in \{1, \dots m\}$ such that $(T - \lambda_i I)$ is not injective

$\Rightarrow$ λ_i is an eigenvalue of T! $\blacksquare$

'v' is not necessarily the eigenvector of T.

Characteristic Polynomial

Motivation: $Av = \lambda v$ $\quad$ A $n \times n$ matrix

$v \neq 0$

$\Rightarrow (A - \lambda I) v = 0$

$\Rightarrow v \in \operatorname{Ker}(A - \lambda I)$

$\Rightarrow \operatorname{rank}(A - \lambda I) < n$

$\dim(V) = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{image}(T))$

$\dim(\operatorname{Ker}(A - \lambda I)) \geq 1$

$\Rightarrow \det(A - \lambda I) = 0$

Def: The characteristic polynomial of a $n \times n$ matrix A is defined as

$$P_A(t) := \det(A - tI)$$

Example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\det(A - tI) = \det\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} a_{11} - t & a_{12} \\ a_{21} & a_{22} - t \end{pmatrix}$$

$$= (a_{11} - t)(a_{22} - t) - a_{12} a_{21}$$

$$= t^2 - t(a_{11} + a_{22}) + a_{11} a_{22} - a_{12} a_{21}$$

Observations:

- $P_A(t)$ is a polynomial with degree n if A is a $n \times n$ matrix.
- Characteristic polynomials do not depend on the choice of basis.

Proof: Consider A, basis transformation matrix U. Want to check if the characteristic polynomial for matrix A and UAU^{-1} are the same.

$$\det(UAU^{-1} - t\, I)$$

$$= \det(UAU^{-1} - t\, UU^{-1})$$

$= \det(UAU^{-1} - tUU^{-1})$

$= \det(U(A - tI)U^{-1})$

$= \det(U) \cdot \det(A - tI) \cdot \det(U^{-1})$

$= \det(A - tI)$

- The roots of the characteristic polynomial correspond exactly to the eigenvalues of A.
- Over $\mathbb{C}$, the charactistic polynomial always has n roots, so the matrix has "n eigenvalues" (not necessarily unique).
- A is invertible $\iff$ 0 is not an eig. val.

If 0 is an eig. val. $A\underline{v} = 0 \cdot \underline{v} = 0$, $v \neq 0$

$\Rightarrow$ Ker(A) is non-trivial $\iff$ A not invertible.

- Let $A \in \mathcal{L}(V)$, λ is a eig. val. of A. Then λ^k is an eig. val. of A^k.

  ↳ geometrically applying A twice will stretch the eig. vec. twice: $\lambda \cdot \lambda = \lambda^2$

- Let A be invertible, λ be eig. val. of A. Then $1/\lambda$ is an eig. val of A^{-1}

  ↳ geometrically inverse is unscaling $\to 1/\lambda$

Def: For an operator A with eigenvalue λ, we define its geometric multiplicity as the dimension of the corresponding eigenspace $E(\lambda, A)$.

Def: The algebraic multiplicity is the multiplicity of the root λ in the characteristic polynomial.

In general, these two notions are not the same

Computing Eigenvalues and Eigenvectors

- Write down the characteristic polynomial, find the roots $\to$ eigenvalues.

- To compute eigenvectors, solve the linear system: $Ax = \lambda x$

Trace of a Matrix

Def: The trace of a square matrix $A \in F^{n \times n}$ is the sum of its diagonal elements: $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$

Remarks:

- $\operatorname{tr}: \mathbb{R}^{n \times n} \to \mathbb{R}$ is a linear operator in particular, $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$
- $\operatorname{tr}(A \cdot B) = \operatorname{tr}(B \cdot A)$

  ⚠ $\operatorname{tr}(A \cdot B) \neq \operatorname{tr}(A) \cdot \operatorname{tr}(B)$
- trace does not depend on the choice of basis.

  Let $T \in \mathcal{L}(V)$, and U and W be two bases of V. Then:

$$\operatorname{tr}(M(T, U)) = \operatorname{tr}(M(T, W))$$

· the trace of an operator equal to the sum of its complex eigenvalues.

$$\tilde{A} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ w.r.t some basis $v_1, v_2, \dots v_n$

$$tr(\tilde{A}) = \sum_{i=1}^{n} \lambda_i$$

Over $\mathbb{C}$, we can always find basis of eigenvectors : $A \in \mathbb{R}^{n \times n}$, over $\mathbb{C}$

I can find representation:

$$\tilde{A} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \lambda_i \in \mathbb{C}$$

$$tr(\tilde{A}) = \sum \underbrace{\lambda_i}_{\in \mathbb{C}} \underset{\substack{\uparrow \\ \text{indep. of} \\ \text{basis.}}}{=} \sum_{i=1}^{n} \underbrace{a_{ii}}_{\in \mathbb{R}} = \underbrace{tr(A)}_{\in \mathbb{R}}$$

$$\Rightarrow \sum \lambda_i \in \mathbb{R}$$

- trace equals the negative of the coefficient corresponding to the $(n-1)$ degree term of the characteristic polynomial.

$$P_A(t) = t^n + a_{n-1}t^{n-1} + \dots\dots$$

- $tr(A)$ = sum of its eig. vals (if exist)

$\det(A)$ = product of its eig. vals. (if exist)

Example: Consider a rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- $R(\theta)$ does not have any real eigenvalues.
- The trace of $R(\theta)$ is $2\cos\theta$
- The characteristic polynomial of $R(\theta)$

is: $P_{R(\theta)}(t) := \det(R(\theta) - tI)$

$$= \det \begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix}$$

$$= (\cos\theta - t)^2 + \sin^2\theta$$

$$= t^2 - 2\cos\theta \cdot t + \underbrace{\cos^2\theta + \sin^2\theta}_{1}$$

$$= t^2 - 2\cos\theta \cdot t + 1$$

- The roots of the characteristic polynomial

$$\lambda_{1/2} = \frac{2\cos\theta \pm \sqrt{(2\cos\theta)^2 - 4}}{2} \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \cos\theta \pm i\sin\theta$$

- The matrix has a diagonal representation

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\operatorname{tr}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$