Probability Measure

- Given space Ω ("abstract space")

- Need a σ-algebra $\mathcal{A}$ on Ω. ("measurable events")
  - $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
  - $(A_i)_{i \in \mathbb{N}} \subset \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ ("countable unions")
  - $\phi, \Omega \in \mathcal{A}$
  - countable intersections

- A measure μ on $(\Omega, \mathcal{A})$ is a function

$$\mu : \mathcal{A} \to [0, \infty]$$

that is countably additive: If $(A_i)_{i \in \mathbb{N}}$ is a sequence of pairwise disjoint sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

A measure P on a measurable space $(\Omega, \mathcal{A})$ is called a probability measure if $P(\Omega) = 1$.

The elements of $\mathcal{A}$ are called events.

Then $(\Omega, \mathcal{A}, P)$ is called a probability space.

Example (1): Throw a die

$\Omega = \{1, 2, \dots, 6\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ (σ-algebra generated by the "elementary events" $\{1\}, \{2\} \dots \{6\}$).

P can be defined uniquely by assigning

$$P(\{1\}) = P(\{2\}) = \dots\dots = P(\{6\}) = 1/6$$

For example $P(\{1,5\}) = P(\{1\}) + P(\{5\}) = 1/3$

Throw two dice:

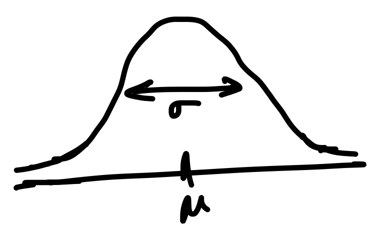
$$\Omega = \{1, 2, \dots, 6\} \times \{1, 2, \dots 6\} = \{\underset{\text{first die}}{(1}, \underset{\text{second die}}{1)}, (1,2) \dots \}$$

elementary events

$$\mathcal{A} = \mathcal{P}(\Omega)$$

$$P(\{(i,j)\}) = 1/36$$

Example (2): Normal distribution

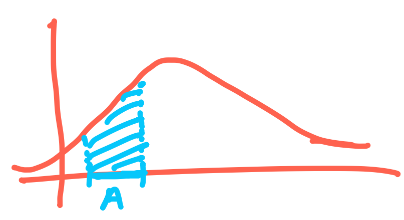


$\Omega = \mathbb{R}$,

$\mathcal{A}$ = Borel-σ-algebra

$f_{\mu,\sigma} : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



$$P : \mathcal{A} \to [0,1] \quad , \quad P(A) := \int_A f_{\mu,\sigma}(x)\, dx$$

Different Types of Probability Measures

Discrete measure:

$\Omega = \{x_1, x_2, \dots\}$ finite & countable

$\mathcal{A} = \mathcal{P}(\Omega)$

We define a probability measure $P: \mathcal{A} \to [0,1]$ by assigning probabilities to the "elementary events":

$$P(\{x_i\}) =: p_i$$

with $0 \le p_i \le 1$, $\sum_i p_i = 1$

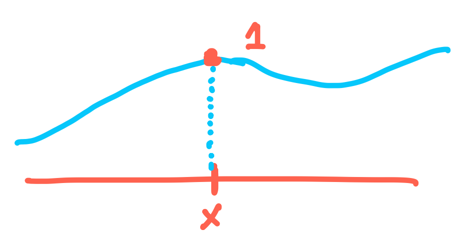
For $A \in \mathcal{A}$ we assign

$$P(A) = \sum_{\{i \mid x_i \in A\}} p_i \, .$$

Examples: a coin toss, distribution on $\mathbb{Q}$

Dirac measure:

For $x \in \mathbb{R}$, we define the Dirac measure δ_x on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by setting



$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Sometimes this is called a point mass at a point x.

A discrete measure on $\mathbb{R}$ can be written as a sum of Dirac measures. For example, throwing a die can be considere as

$$\frac{1}{6}\left(\delta_1 + \delta_2 + \dots + \delta_6\right)$$

Measures with a density

Consider $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and the Lebesgue measure λ. Consider a function $f: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is measurable and satisfies $\int f \, d\lambda = 1$

$$\Downarrow$$

$$\int f(x)\, dx$$

Then we define a measure γ on $\mathbb{R}^n$ by setting, for all $A \in \mathcal{A}$,

$$\gamma(A) := \int_A f(x)\, dx$$

γ is the probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with density f.

Notation: $\gamma = f \cdot \lambda$

Question: Can we describe every probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ in terms of a density? Answer: no!

Counterexample : δ_0 Dirac measure.

On the same measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$,
if we have two measures λ, γ.

Question: $\quad \gamma(A) = \int_A \phi \, d\lambda$

does ϕ exist? No!

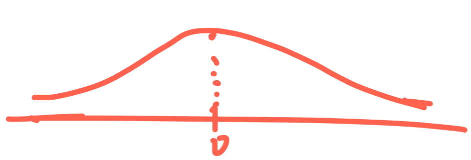
Def: A probability measure γ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is called absolutely continuous with respect to another measure μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ if every μ-null set is also a γ-null set

$$\forall B \in \mathcal{B}(\mathbb{R}^n): \quad \mu(B) = 0 \Rightarrow \gamma(B) = 0.$$

Notation: $\gamma \ll \mu$

$$\mu(A) = 0 \Rightarrow \underbrace{\int_A f \, d\mu}_{\gamma(A)} = 0$$

Example: $N(0,1) \ll \lambda$



$$\gamma_A = \int_A f \, d\lambda$$

Example: $\delta_0 \not\ll \lambda$ because

$$\lambda(\{0\}) = 0 \quad \text{but} \quad \delta_0(\{0\}) = 1$$

Theorem (Radon-Nikodym): Consider two prob. measures γ, μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then the following two statements are equivalent:

(1) γ has a density w.r.t. μ.

(2) γ is absolutely continuous w.r.t μ.

If $\gamma \ll \mu$, then $\exists\ \phi$ such that $\gamma(A) = \int_A \phi \, d\mu$

ϕ is exists and is unique.

Proof idea:

(1) $\Rightarrow$ (2) easy

(2) $\Rightarrow$ (1) We need to construct a density!

Consider the set $\mathcal{G}$ of all functions g with the following properties:

$$
\circledast \begin{cases} \cdot\ g \text{ is measurable}, \quad g \geqslant 0 \\ \cdot\ g \cdot \mu \leq \gamma, \text{ that is} \\ \qquad \forall A \in \mathcal{B}(\mathbb{R}^n) : \int_A g \, d\mu \leq \gamma(A). \end{cases}
$$

- Observe: $g \equiv 0$ satisfies $\circledast$, so $\mathcal{G}$ is not empty.
- If g, h both satisfy $\circledast$, then $\sup(g, h)$ satisfies $\circledast$.
- Define $\varkappa := \sup_{g \in \mathcal{G}} \int g \, d\mu$ and construct a sequence $(g_n)_{n \in \mathbb{N}}$ such that $\lim \int g_n \, d\mu = \varkappa$.
- Define "density" $f := \sup g_n$
- Now prove: f is the density that we are looking for.

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